MCS 224 2011-2012 Spring
Exercise Set I

1. Six married couples are standing in a room.
   (a) If 2 people are chosen at random, find the probability p that (a) they are married, (b) one is male and one is female.
   (b) If 4 people are chosen at random, find the probability p that (a) 2 married couples are chosen, (b) no married couple is among the 4, (c) exactly one married couple among the 4.
   (c) If the 12 people are divided into six pairs, find the probability p that (a) each pair is married, (b) each pair contains a male and a female.
2. A point is selected at random inside a circle. Find the probability p that the point is closer to the center of the circle than to its circumference.
3. Two digits are selected at random from the digits 1 through 9. If the sum is even, find the probability p that both numbers are odd.
4. Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn. If the number is even, find the probability that the card came from box A.
5. An urn contains 3 red marbles and 7 white marbles. A marble is drawn from the urn and a marble of the other color is then put into the urn. A second marble is drawn from the urn.
   (a) Find the probability p that the second marble is red.
   (b) If both marbles were the same color, what is the probability p that they were both white?
6. Four persons, called North, South, East and West, are each dealt 13 cards from an ordinary deck of 52 cards.
   (a) If South has exactly one ace, what is the probability that his partner North has the other three aces?
   (b) If North and South together have 10 hearts, what is the probability that either East or West has the other 3 hearts?
7. A fair coin is tossed three times. Let X denote 0 or 1 according as a head or a tail occurs on the first toss, and let Y denote the number of heads which occur. Determine
   (a) the joint probability distributions of X and Y,

(b) the marginal distributions of X and Y.

8. Let X be a continuous random variable with distribution

$$f(x) = \begin{cases} \frac{1}{6}x + k & \text{if } 0 \leq x \leq 3, \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Evaluate k.

   (b) Find $P(1 \leq X \leq 2)$.

9. The probability of team A winning any game is $\dfrac{1}{2}$. A plays team B in a tournament. The first team to win 2 games in a row or a total of three games wins the tournament. Find the expected number of games in the tournament.

10. Suppose that n indistinguishable balls are distributed into n boxes, so that each distinguishable arrangement equally likely. Find the probability that no box is empty.

11. If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of false alarm (a false indication of aircraft presence), and the probability of missed detection (nothing registers, even though an aircraft is present)? What is the probability that an aircraft is present given that the radar registers an aircraft presence?

12. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4. What is the probability that each group includes a graduate student?

13. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning? Suppose that you win. What is the probability that you had an opponent of type 1?

14. An integer number randomly selected between numbers 100 and 999, inclusive.

    (a) What is the probability that selected number has at least one "5" in it?

    (b) What is the probability that there are exactly two "5"s in it?

15. Suppose n distinguishable balls are distributed at random into r boxes. Find the probability that there are exactly k balls in the first box.

16. Urn 1 contains 4 red and 6 green balls while urn 2 contains 6 red and 3 green balls. A ball is selected at random from urn 1 and transferred to urn 2. Then a ball is selected at random from urn 2. Given that the ball from urn 2 is green, find the conditional probability that the ball from urn 1 was green?

17. Consider the experiment of rolling two dice. Define the events A, B, C so that

$$A = \{\text{First die results in a 1, 2, or 3}\}$$
$$B = \{\text{First die results in a 3, 4, or 5}\}$$
$$C = \{\text{The sum of two faces is 9}\}$$

Show that A, B and C are not pairwise independent but

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

18. Suppose that 5% of men 2.5% of women are colorblind. A colorblind person is chosen at random, assume that there are equal number of males and females, what is the probability of this person being male? What would be your answer to the question if the number of males is the twice of the number of females in the population?

19. Two construction contracts are to be randomly assigned to one or more of 3 firms: I, II, and III. Any firm may receive more than one contract. If each contract will yield a profit of 90 000 dollars, find the expected profit for firm I. If firms I and II are actually owned by the same individual, what is the owner's expected total profit?

20. The game of "craps" is played as follows. A pair of fair dice is rolled. If the sum is 7 or 11 you win, if the sum is 2, 3, or 12 you lose. With any other sum, you continue to roll until you roll the initial number again (then you win) or you roll a 7 (in which case you lose). Let X be the r.v and $X = 0$, if you lose on the first roll, $X = 1$, if you win on the first roll and $X = 2$ if you continue to roll the dice. Find the cumulative distribution function of X.

21. If random variable X has the distribution function

$$F_X(t) = \begin{cases} 0 & x < 1 \\ \frac{1}{3} & 1 \leq x < 4 \\ \frac{1}{2} & 4 \leq x < 6 \\ \frac{5}{6} & 6 \leq x < 10 \\ 1 & x \geq 10 \end{cases}$$

find

(a) $P(2 < X \leq 6)$,

(b) $P(X = 4)$,

(c) the probability distribution of X,

(d) the expected value of X,

(e) the variance of X.

22. An urn contains 10 balls of which 1 is black. Let X be the number of draws, with replacement, necessary to observe the black ball. Find the probability function and the mean of X.

23. The probability that a player wins a game at a single trial is $\frac{1}{3}$. If the player plays until he wins, assuming the independency of these trials, find the probability that the number of trials (until he wins) is divisible by 4.

24. Out of a job population of 10 jobs with 6 jobs of class 1 and 4 of class 2, a random sample of size 5 is selected, without replacement. Let X be the number of class 1 jobs in the sample. Determine the probability distribution function of X. Find μ_X and σ_X^2.

25. Consider a continuous random variable X with density

$$f(x) = \begin{cases} ce^x & \text{if} \quad x < 0, \\ 0 & \text{if} \quad \text{elsewhere.} \end{cases}$$

   (a) Find the value of constant c.
   (b) Find the mean of X.
   (c) Find the cumulative distribution function of X.

26. Two electronic components of a missile system work in harmony for the success of the total system. Let X and Y denote the life in hours of the two components. The joint density of X and Y is

$$f(x, y) = \begin{cases} ye^{-y(1+x)} & \text{if } x, y \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Give the marginal density functions for both random variables.
   (b) What is the probability that both components will exceed 2 hours?

27. A chemical system that results from a chemical reaction has two important components among others in a blend. The joint distribution describing the proportion X_1 and X_2 of these two components is given by

$$f(x_1, x_2) = \begin{cases} 2 & \text{if } 0 < x_1 < x_2 < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

   (a) Give the marginal distribution of X_1.
   (b) Give the marginal distribution of X_2.
   (c) What, is the probability that component proportions produce the results $X_1 < 0.2$ and $X_2 > 0.5$?
   (d) Give the conditional distribution $f(x_1|x_2)$.

28. Consider the random variables X and Y that represent the number of vehicles that arrive at 2 separate street corners during a certain 2-minute period. These street corners are fairly close together so it is important that traffic engineers deal with them jointly if necessary. The joint distribution of X and Y is known to be

$$f(x, y) = \frac{9}{16} \frac{1}{4^{(x+y)}}, \quad x = 0, 1, 2, \ldots \text{ and } y = 0, 1, 2, \ldots .$$

   (a) Are the two random variables X and Y independent? Explain why or why not.
   (b) What is the probability that during the time period in question less than 4 vehicles arrive at the two street corners?